A

Research Project

on

DYNAMICS OF THE LORENZ CURVE

submitted by

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under the supervision of

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TITLE OF RESEARCH PROJECT

DYNAMICS OF THE LORENZ CURVE

Contents

ABSTRACT

In this work, we explore dynamic systems through the lens of the Lorenz system. Briefly mention the significance of the Lorenz system in chaos theory. Firstly, we begin with "The Lorenz System". We introduce motivation and introduction of Lorenz System. And we include the definition of some important terms that are related to the topic. Secondly, we learn about "Bifurcation". We figure out the introduction and bifurcation diagram. We try to find the relation between bifurcation and the Lorenz system. Thirdly, we work on "Sensitivity to Initial Conditions" which talks about its introduction and emphasizes the exploration of sensitivity to initial conditions in chaotic systems, with a specific focus on the Lorenz system. Highlight the importance of understanding how small changes can lead to vastly different outcomes, namely we analyze the Lorenz curve. Lastly, we include an "Appendix" that provides Python and Mathematica code which corresponds to the figures.

Keywords: Lorenz system, Bifurcation, and Sensitive to Initial Condition.

AIM/OBJECTIVE

The objectives of this project work are as follows:

- We introduce motivation, the introduction of the Lorenz system, and the definition of some terms that involve the topic.
- We study the bifurcation that includes its introduction, Bifurcation Diagram or Period-Doubling Bifurcation Diagram, and the Relation Between Bifurcation and Lorenz System.
- We find out the introduction of Sensitivity To Initial Conditions and analyze it.

In this project, we aim the following:

- First, we start introducing motivation to get the passion for the topic. After that, we figure out the system of ordinary differential equations by introducing variables and parameters. Also, we include definitions of some related topics.
- Second, we study the analysis of bifurcation. We introduce a bit about it and then we study the relation between bifurcation and Lorenz system and sensitivity to the initial condition.
- Third, we work on the important property of the Lorenz system, which is called sensitive to initial condition. We analyze the Lorenz curve based on the initial condition and the new initial condition. Also, we just detail how to reduce the divergence from the Lorenz curve. In the end, we detail some applications of the analysis of the Lorenz curve.

INTRODUCTION OF RESEARCH TOPIC

The Lorenz System

1. Motivation and Introduction

In the middle of the 1900s, computer, and satellite technology was being developed and it was believed this would allow the human race to completely predict and control the weather. Unfortunately, this has not happened because everyone knows that weather forecasting is still not very accurate. The problem was the assumption that tiny perturbations in the system only amount to tiny changes over time.

In 1963, **Edward Lorenz** (meteorologist and mathematician) published his paper, *Deterministic Nonperiodic Flow*, in which he showed that tiny differences in the initial conditions amount to dramatic differences in the systems' behavior over time. He, with the help of **Ellen Fetter** who was responsible for the numerical simulations and figures, and **Margaret Hamilton** who helped in the initial, numerical computations leading up to the findings of the Lorenz model, developed a simplified mathematical model for atmospheric convection.

The motivation behind the development of the Lorenz system of dynamics, specifically chaos theory, lies in Edward Lorenz's desire to understand the inherent complexities and difficulties in predicting weather patterns. In the early 1960s, Lorenz was working on developing a mathematical model for atmospheric convection, aiming to improve weather forecasting.

Lorenz started with a set of simplified equations representing fluid dynamics and heat transfer in the atmosphere. As he refined his model, he discovered that even small changes in the initial conditions of the system could lead to vastly different outcomes over time. This sensitivity to initial conditions was a groundbreaking revelation and marked the birth of chaos theory.

The Lorenz system of dynamics is a set of three coupled ordinary differential equations that describe the behavior of a simplified model of atmospheric convection. It is known for its chaotic solutions, which means that even small variations in the initial conditions can lead to dramatically different trajectories over time. This sensitivity to initial conditions is a hallmark of chaotic systems and is often referred to as the "butterfly effect," emphasizing the idea that the flap of a butterfly's wings could potentially influence the weather on a larger scale. The system is known for its chaotic behavior, which means that small changes in the initial conditions can lead to drastically different outcomes over time.

The three-dimensional Lorenz system is given by the following set of equations:

$$\begin{aligned}\frac{dx}{dt} &= \sigma(y - x), \\ \frac{dy}{dt} &= x(\rho - z) - y, \\ \frac{dz}{dt} &= xy - \beta z\end{aligned} \tag{1.2.1}$$

Here, x, y, and z represent the variables of the system that are associated with convective motion, related to the convective motion, and associated with temperature differences respectively. While σ, ρ and β are parameters that can be adjusted to observe different behaviors and they are system parameters proportional to the Prandtl number, Rayleigh number, and certain physical dimensions of the layer itself. The variable t represents time, and $\frac{dx}{dt}$, $\frac{dy}{dt}$ and $\frac{dz}{dt}$ represent the rate of change of the variables with respect to time, respectively.

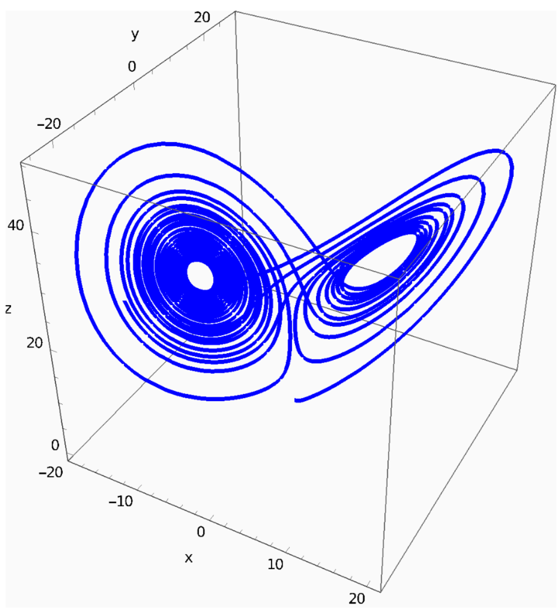

Figure 1: The classic "Lorenz Butterfly" showing the genericc shape of all solutions with parameter $\sigma = 10$, $\rho = 38$, $\beta = \frac{8}{3}$

2. Definition of Some Terms

Chaotic means that in a system, even tiny changes in the starting conditions can lead to very different outcomes over time. It describes behavior that seems random and hard to predict, yet follows specific rules.

Attractor is like a destination in a dynamic system where the system tends to settle over time. In chaos, the attractor can be a complex and repeating pattern, drawing trajectories toward it.

Equilibrium Point is a state where the system doesn't change over time. In chaotic systems, equilibrium points are often unstable, meaning small disturbances can lead to chaotic behavior.

Beyond Chaos refers to the unpredictable nature of systems with chaotic behavior. It suggests that, over time, the system's evolution becomes so intricate that precise predictions become nearly impossible.

Periodic Orbit is a repeating pattern in the system's behavior. In chaos, periodic orbits may exist within the chaotic dynamics, contributing to the complex and varied trajectories.

Butterfly Effect is the idea that a small change in one part of a system can lead to significant consequences in another part. It's a metaphor for the extreme sensitivity to initial conditions in chaotic systems.

THEORETICAL FRAMEWORK OF TOPIC

Bifurcation

1. Introduction

Bifurcation in dynamic systems is a captivating phenomenon that unveils the complex and often unexpected behavior of nonlinear systems as parameters are varied. It marks critical points where the qualitative nature of the system's solutions undergoes a transformative change. The study of bifurcations is foundational in understanding the emergence of order, chaos, and complexity in diverse fields such as physics, engineering, biology, and economics.

At its core, bifurcation is a mathematical concept that describes how the equilibrium or periodic solutions of a dynamical system evolve as parameters undergo changes. It provides insights into the stability and qualitative features of solutions in response to alterations in the system's governing parameters.

The four types of bifurcations are:

1. Pitchfork Bifurcation
2. Hopf Bifurcation
3. Transcritical Bifurcation
4. Limit Point (Saddle Node) Bifurcation.

In this work, we will not figure out the types of bifurcations, namely we focus on interpreting the processing of the Period-Doubling Bifurcation Diagram and the relation between bifurcation and the Lorenz system or sensitivity of the initial condition of the Lorenz system.

2. Bifurcation Diagram or Period-Doubling Bifurcation Diagram

The primary parameters influencing the system are denoted as σ, ρ, and β. Bifurcation analysis involves systematically varying these parameters to observe the resulting changes in the system's behavior. One of the most notable bifurcations in the Lorenz system is the period-doubling bifurcation. Period-doubling bifurcation is a type of bifurcation in which the periodicity of a system's behavior doubles as a control parameter is varied. As a bifurcation parameter, often ρ is increased, the system undergoes a series of period-doubling events leading to a cascade of bifurcations and the emergence of chaotic behavior.

Consider the bifurcation diagram or period-doubling bifurcation diagram:

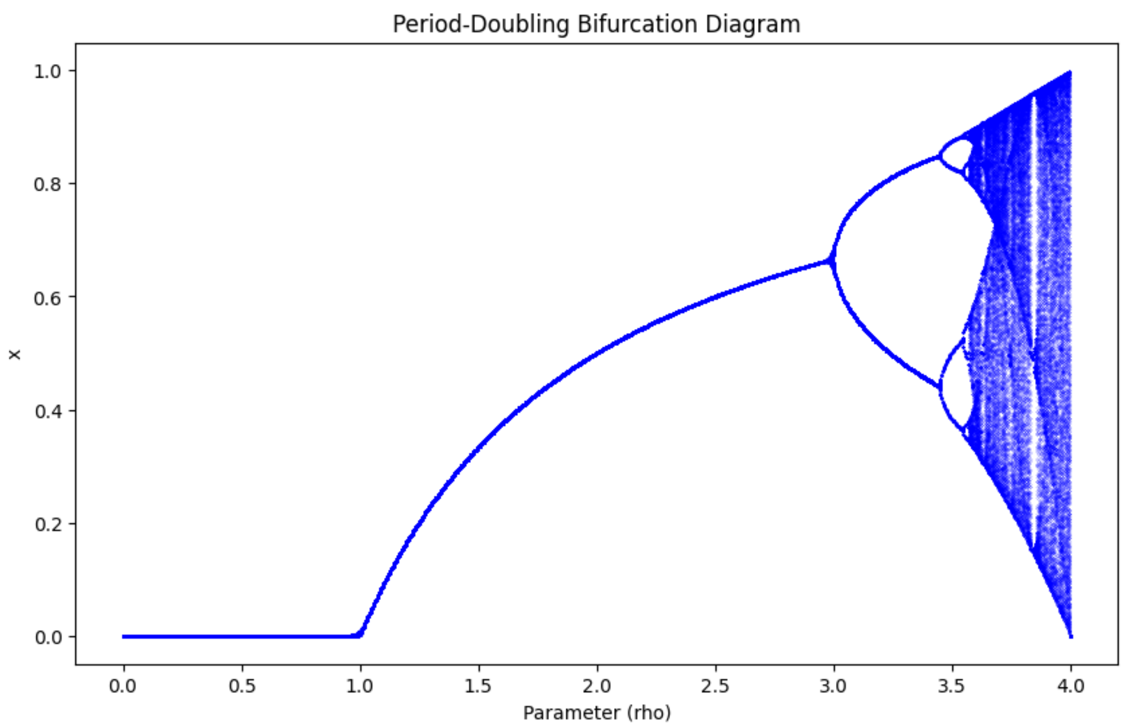

Figure 2: Logistic Map of Period-Doubling Bifurcation Diagram

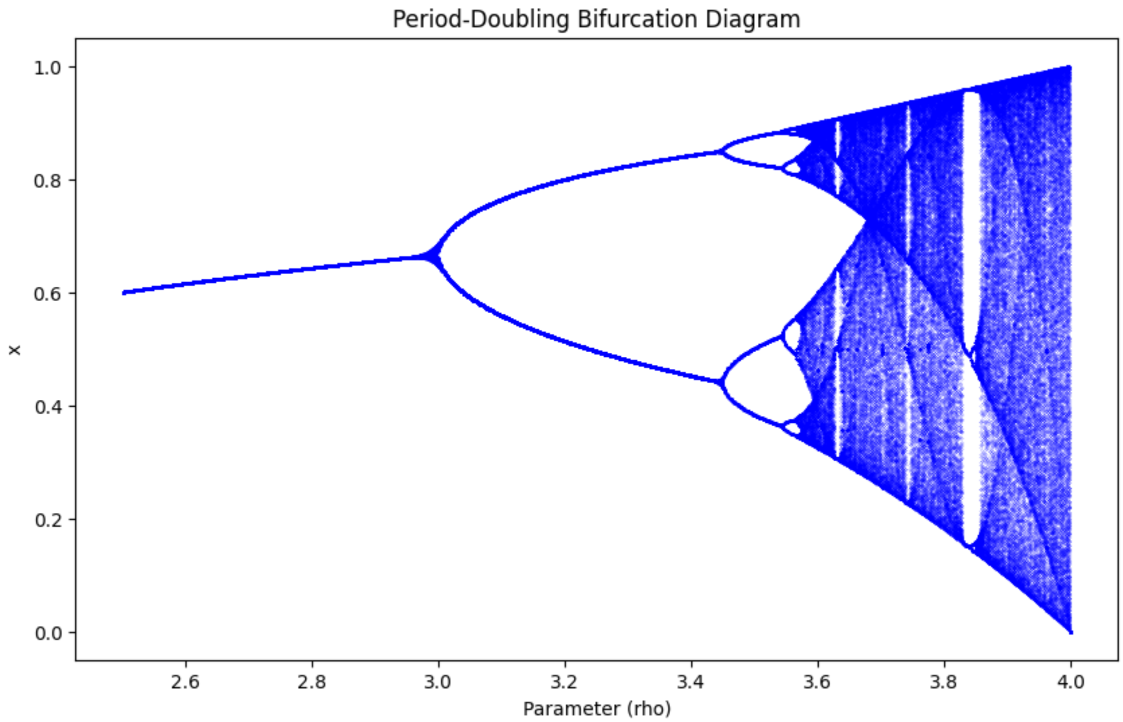

Figure 3: Logistic Map of Period-Doubling Bifurcation Diagram

Observation the figure 2 and 3, we witness a captivating transformation as we tweak the parameter ρ. Here's the breakdown

- For $\rho < 1$: All points converge to zero, indicating a stable state with a single-point attractor, equilibrium, emphasizing the simplicity of the system.
- For $1 < \rho < 3$: The system maintains one-point attractors, but the 'attracted' value of x increases gradually with ρ.
- Bifurcations: At specific ρ values like 3, 3.45, 3.54, the system undergoes bifurcations, doubling the periodicity of the orbits.
- Chaos at $\rho \approx 3.57$: Just beyond this point, chaos emerges. The diagram shows irregular patterns, signifying unpredictability.
- Beyond Chaos: Surprisingly, chaos isn't constant for all $\rho > 3.57$. The system oscillates between chaos and stability, creating a dynamic landscape.

3. Relation Between Bifurcation and Lorenz System

From part 2, we will continue to figure out the relation between the bifurcation diagram and the Lorenz system. In the bifurcation diagram of the Lorenz system, we dive into a dynamic landscape shaped by the interplay of chaos, stability, and the system's sensitivity to initial conditions.

- Stability at Lower ρ: For lower values of the bifurcation parameter ρ, the system exhibits stable behavior. Trajectories converge to fixed points or periodic orbits, revealing a sense of order.

- Bifurcations and Complexity: As ρ increases, the bifurcation diagram unveils bifurcation points. These points mark transitions from stable behavior to more complex dynamics. Successive bifurcations introduce new periodic orbits, leading to intricate patterns in the diagram.

- Sensitive Dependence on Initial Conditions: Notably, the Lorenz system demonstrates sensitivity to initial conditions. Small changes in the starting conditions can result in drastically different trajectories. This sensitivity is visualized in the diagram as nearby trajectories diverge over time, highlighting the system's chaotic nature.

- Chaos Emergence: Beyond certain critical values of ρ chaos emerges. Chaotic regions in the diagram are characterized by a lack of long-term predictability, and trajectories exhibit a sensitive dependence on initial conditions. Even slight variations in the starting state lead to significantly different outcomes.

- Intermittent Stability: Intriguingly, amidst chaotic regimes, pockets of stability may appear. These regions suggest temporary returns to more ordered behavior before the system reenters chaotic dynamics.

METHODOLOGY FOLLOWED

Sensitivity To Initial Conditions

1. Introduction

Sensitivity to initial conditions is a fascinating concept that reveals the unpredictable nature of certain dynamic systems. In these systems, even tiny variations in the starting conditions can lead to vastly different outcomes over time. This sensitivity, often referred to as the "butterfly effect," is a hallmark of chaotic behavior.

Imagine a scenario where the initial state of a system, such as the position and velocity of particles or the atmospheric conditions, determines its future trajectory. Sensitivity to initial conditions means that small uncertainties or changes in this starting state can amplify over time, causing trajectories to diverge in unexpected ways.

This phenomenon is particularly evident in chaotic systems, where deterministic equations govern the evolution of the system, yet the outcomes appear inherently unpredictable. The sensitivity to initial conditions introduces an element of complexity and challenges our traditional notions of predictability.

Studying sensitivity to initial conditions not only unveils the intricacies of chaotic systems but also has practical implications, from weather forecasting to understanding the dynamics of complex phenomena. As we explore this concept, we delve into the delicate balance between order and chaos, where a minute change in the initial state can lead to a cascade of unpredictable events, underscoring the richness and complexity of dynamic systems.

2. Analysis On Sensitive To Initial Conditions

Consider the Lorenz system (1.2.1)

$$\frac{dx}{dt} = \sigma(y - x),$$
$$\frac{dy}{dt} = x(\rho - z) - y,$$
$$\frac{dz}{dt} = xy - \beta z$$

Finding an analytical solution for the Lorenz system is hard due to its complex, nonlinear nature and chaotic behavior. The equations involve terms that don't easily lead to simple solutions. Additionally, small changes in the initial conditions can result in vastly different outcomes, making it challenging to express a precise solution. Numerical methods are often used such as Runge-Kutta methods or adaptive-step methods, can be employed to approximate solutions, but even they can face difficulties in chaotic situations. The Lorenz system's iconic status stems from its role in illustrating the intricate dynamics of chaos rather than offering easily obtainable analytical answers. If the reader wants to generate an answer to the Lorenz system, you can get coding in Appendix and generate them to get an answer.

Assume the solution of the Lorenz system happens with initial condition (x_0, y_0, z_0) and parameters

$$\sigma = 10,\ \rho = 28,\ \text{and}\ \ \beta = \frac{8}{3}.$$

Now, a slight change in only one component in the initial conditions namely

$$(x_0, y_0, z_0) \quad \text{to be} \quad (x_0 + \epsilon, y_0, z_0)\,,$$

where

$$\epsilon = 0.0000000000001.$$

By taking original initial condition

$$(x_0, y_0, z_0) = (0, 1, 1)$$

and new initial condition

$$(x_0 + \epsilon, y_0, z_0) = (0.0000000000001, 1, 1)$$

Then, observe the phase portait for variable x, y and z respect to t respectively where label red curve is corresponding with (x_0, y_0, z_0) and blue curve is corresponding with $(x_0 + \epsilon, y_0, z_0)$.

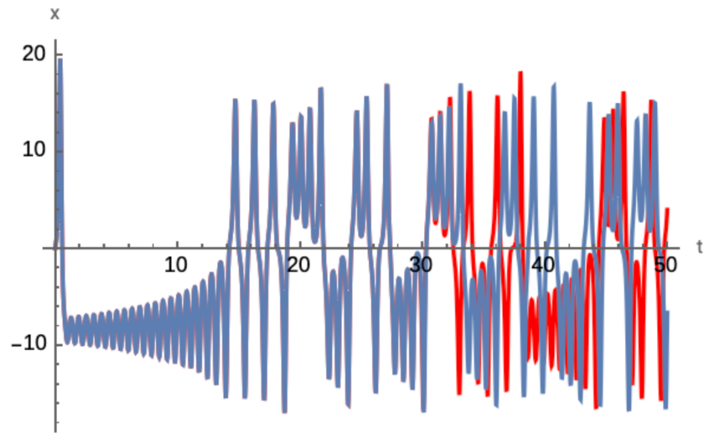

Figure 4: Phase portait for variable x respect to t

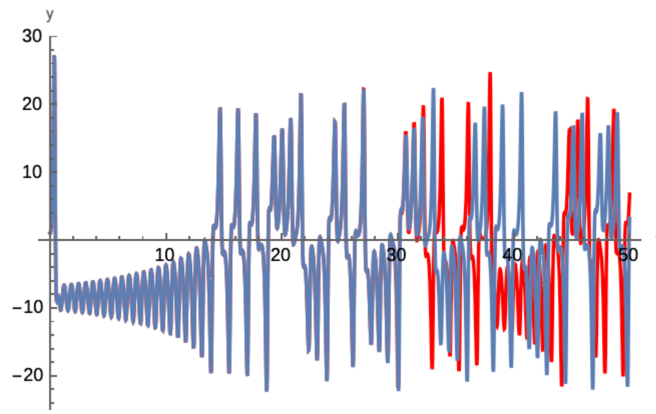

Figure 5: Phase portait for variable y respect to t

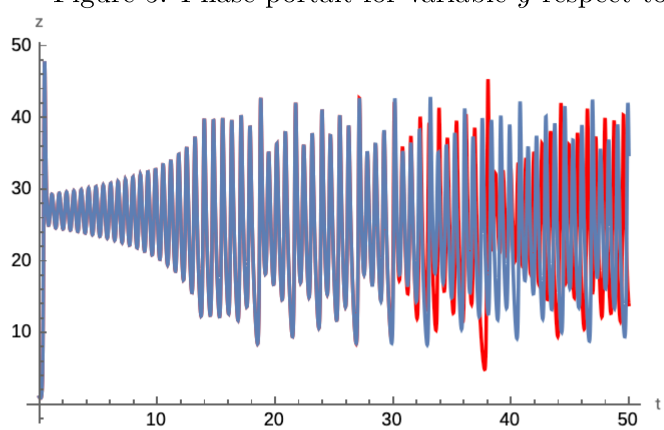

Figure 6: Phase portait for variable z respect to t

Now, let's interpret the meaning of the resulting curve:

- Interpretation: If the system is highly sensitive to initial conditions, you will observe significant divergence between the red and blue trajectories over time. The phase portrait visually demonstrates how a minute change in the starting state leads to a distinct and potentially divergent evolution of the system.

- Sensitivity to Initial Conditions: The Lorenz system is known for its sensitivity to initial conditions. Even a small difference in the initial conditions can lead to significantly different trajectories over time. This sensitivity is a hallmark of chaotic systems.

- Comparison of Trajectories: By plotting the solutions for two slightly different initial conditions, the curve illustrates how the trajectories move away over time.

- Chaotic Dynamics: Chaotic behavior is observed when the trajectories exhibit complex, non-repeating patterns. The overlapping or diverging nature of the curves in the plot is indicative of chaotic dynamics in the Lorenz system.

- Butterfly Effect: The butterfly effect is evident in the curve, showcasing how a small change in the initial conditions can lead to significant differences in the evolution of the system. This is a fundamental characteristic of chaotic systems like the Lorenz system.

- No Predictability Beyond a Point: After a certain time, it would become impossible to predict the state of one trajectory based on the other, despite their close initial states.

- Some attractors: while the exact path each trajectory takes will be different, they will be different, they will both be bounded within the same general shape in phase space.

- Loss of information: If you used one trajectory to try and backtrack to the initial conditions, after a certain point, it would become nearly impossible to determine the exact initial state because so many different initial conditions would have evolved to a similar state.

In summary, the plot provides a visual representation of the sensitivity to initial conditions and chaotic dynamics in the Lorenz system. It illustrates how small perturbations in the starting state lead to divergent trajectories over time, emphasizing the intricate and unpredictable nature of chaotic systems.

In chaotic systems like the Lorenz system, trajectories inherently diverge due to their sensitivity to initial conditions. This sensitivity is a fundamental characteristic of chaotic behavior and is challenging to eliminate. However, if you want to make trajectories less divergent:

- Stabilization Techniques: Explore control and stabilization methods to reduce sensitivity, but complete elimination of divergence may not be achievable.
- Adjust Parameters: Experiment with changing system parameters to find regimes with more stable behavior.
- Resonance Islands: Certain parameter combinations may lead to periodic orbits or resonance islands within chaos, offering more predictability.
- Controlled Perturbations: Introduce controlled, small perturbations to understand system responses without trying to eliminate divergence.
- Numerical Precision: Use high-precision numerical methods to minimize errors that can contribute to divergence.
- Explore Bifurcation Diagrams: Study bifurcation diagrams to identify parameter values where the system exhibits stable behavior.
- Time-Averaging: Consider averaging techniques to study statistical properties over time, providing a more stable representation.

If the system has a stable equilibrium point and the initial conditions are changed within the basin of attraction of that equilibrium, the trajectory will still converge to the same equilibrium. It's important to note that eliminating divergence in chaotic systems is generally not possible, as it is a fundamental property of chaotic behavior. The sensitivity to initial conditions is what characterizes chaos. The suggested approaches may help you explore stability or find specific conditions where trajectories are less divergent, but full stability in chaotic systems remains a challenging and nuanced topic.

ANALYSIS AND INTERPRETATION OF RESEARCH PROJECT

Application of Analyzing Lorenz Curve

Analyzing the Lorenz curve in the context of dynamic systems, especially chaotic systems like the Lorenz system, offers several advantages in gaining insights into the behavior and characteristics of the system.

Here are some advantages:

- Visualization of Chaos: The Lorenz curve provides a visual representation of the chaotic behavior of trajectories in phase space. It allows researchers and analysts to observe the intricate and complex patterns that characterize chaos.

- Sensitivity Analysis: By comparing Lorenz curves for different initial conditions or parameter values, sensitivity to perturbations and variations can be assessed. This is crucial for understanding how small changes in the system's state or parameters lead to divergent trajectories.

- Identification of Bifurcations: Changes in the structure of the Lorenz curve can signal bifurcations in the system. Bifurcations are critical points where the system undergoes a qualitative change in behavior. Analyzing the Lorenz curve aids in identifying and characterizing these transitions.

- Quantitative Assessment of Chaos: The Lorenz curve provides a quantitative measure of the chaotic nature of the system. Complex and irregular Lorenz curves are indicative of chaotic behavior, and specific features of the curve can be analyzed to quantify the degree of chaos.

- Stability Assessment: The stability of the system can be assessed by studying the evolution of the Lorenz curve. Stable systems may exhibit more consistent curves, while unstable or chaotic systems may show irregularities, fluctuations, or bifurcations.

- Parameter Tuning: Researchers can use the Lorenz curve to explore and tune parameters in the system. It helps in understanding how changes in parameters affect the distribution of states and can guide parameter selection for desired behaviors.

- Prediction of Short-Term Trends: While chaotic systems are inherently unpredictable in the long term, analyzing the Lorenz curve can offer insights into short-term trends and patterns. This can be valuable for short-term forecasting and understanding the immediate evolution of the system.

- Complexity Analysis: The Lorenz curve aids in assessing the complexity of the system. Complex, non-linear dynamics often manifest as intricate Lorenz curves, and analyzing their features can provide a quantitative measure of the system's complexity.

- Insights into Stochastic Resonance: The Lorenz curve can be utilized to study stochastic resonance in chaotic systems. Stochastic perturbations can impact the shape of the Lorenz curve, providing insights into how the system responds to external stimuli.

In summary, the analysis of the Lorenz curve in dynamic systems offers a range of advantages, from visualizing chaos to quantifying sensitivity, stability, and complexity. It serves as a powerful tool for understanding the intricate dynamics of chaotic systems.

PROFILE OF ORGANIZATION/RESEARCH LAB

Kirori Mal College, an institution of academic excellence, established in 1954, that has always strived to, and successfully maintained its place as one of the finest within the University of Delhi. Kirori Mal College believe in providing the students an environment which is rich in knowledge and supportive of their extracurricular interests. The college encourages a quest for knowledge that is rooted in an ethical understanding of the world that we inhabit and this enthusiasm for learning. Kirori Mal College fosters an atmosphere of intellectual vigor and moral rectitude in which the youth may find their fulfillment and achieve greatness as eminent citizens.

- **Mathematics laboratory:** There is a well equipped Mathematics Laboratory assigned by the College to the Department of Mathematics. In order to cater to the needs of the individual differences among students, hands on training and practical exposure/experiences is given to the students to understand the Mathematical concept. Gone are the days when mathematics was treated a purely theoretical subject to be discussed only on the black board or slates. It has now been realized that its nature is as practical as other sciences or technical subjects. Use of advanced and updated mathematical software like MS-Office, Mathematica, and MathType to typeset in Mathematics are present in the laboratory which help students to explore more possibilities about Mathematics.

- **College library:** It has a collection of several books which is beneficial for our students. The college library has been enriched with addition of books. INFLIBNET- NLIST is made available for on-line references which help the teachers and students in their research. Internet and Wi-Fi facilities are provided to all the computers in the college.

CONCLUSION AND SUGGESTION OF RESEARCH PROJECT

1. Conclusion/Suggestion

To achieve the main objectives we have set for our work, we first have to build up all the basic concepts needed then immediately we go ahead with our main theorems which are followed by some useful applications, for each work.

The Lorenz System: We have to know the basics of ordinary differential equations (ODEs) to understand the math behind the Lorenz system. Since it is very difficult to find the solution for Lorenz systems (non-linear), we should know the essential numerical methods used for solving ODEs and simulating dynamic systems.

Bifurcation Fundamentals: Before we learn the deep meaning of bifurcation, we have to take concepts of bifurcation theory, highlighting its role in understanding shifts in system behavior. And we can make sure to discuss how bifurcation theory is practically applied to dynamic systems, revealing important changes.

Sensitivity to Initial Conditions: we should explore the concept of sensitivity to initial conditions, emphasizing its impact on system trajectories. Also, we try to find out how sensitivity contributes to chaos, laying the groundwork for analysis in the Lorenz curve.

2. Future Work

There are many more interesting parts I am passionate about in the Lorenz system of dynamics. Exciting avenues for future exploration beckon such as

- Higher Dimensions: Extend our analysis to higher dimensions, examining how additional variables influence the Lorenz system's behavior.
- Real-World Applications: Apply insights from the Lorenz system to real-world phenomena, like atmospheric or ecological systems.
- Advanced Computing: Explore advanced numerical techniques or computational methods for more efficient simulations, possibly incorporating machine learning.

RESULT(S) ACHIEVED

This project delves into the fascinating world of Lorenz systems of dynamics, with a primary focus on the analysis of the Lorenz curve. The exploration unfolds as follows:

The Lorenz System: We delve into the origin and characteristics of the Lorenz system. Recognized as a quintessential model in chaos theory, the Lorenz system's chaotic behavior and applications in various fields are examined. This part provides a foundational understanding of the system's significance and its role as a dynamic model.

Bifurcation: We navigates the intricate concept of bifurcation within dynamic systems, with a specific emphasis on its implications for the Lorenz system. Bifurcation analysis is explored as a key tool in unraveling the complex behavior of the system, shedding light on transitions and transformations that contribute to its dynamic nature.

Sensitivity to Initial Conditions: We delves into the crucial aspect of sensitivity to initial conditions, a hallmark of chaotic systems. Centered around the Lorenz system, this section investigates how minute changes in the starting state can lead to profound variations in trajectories of the Lorenz curve. The exploration underscores the importance of understanding and navigating the unpredictable nature of dynamic systems.

Throughout the project, methodologies specific to the Lorenz system are applied to deepen the analysis of the Lorenz curve. The culmination of these efforts reveals key findings that not only contribute to a better comprehension of chaotic systems but also offer insights into the broader field of dynamic systems.

In essence, this project provides a comprehensive examination of the Lorenz system, elucidating its chaotic dynamics, exploring bifurcation phenomena, and unveiling the system's sensitivity to initial conditions. Through this exploration, the project contributes valuable insights to the broader understanding of dynamic systems and chaos theory.

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Appendix

Below is the Mathematica code of figure 1:

```
(*Parameters for the Lorenz system*)
σ = 10;
ρ = 28;
β = 8 / 3;

(*Lorenz system of equations*)
lorenz = {
x '[t] == σ (y[t] - x[t]),
y '[t] == x[t] (ρ - z[t]) - y[t],
z '[t] == x[t] × y[t] - β * z[t]
};
(* Initial condition *)
initialCondition = {x[0] == 0, y[0] == 1, z[0] == 1.05};

(* Numerically solve the system *)
sol = NDSolve[{lorenz, initialCondition}, {x, y, z}, {t, 0, 25}];

(* Plot the solution in 3D space *)
ParametricPlot3D[Evaluate[{x[t], y[t], z[t]} /. sol], {t, 0, 25},
 PlotRange → All, AxesLabel → {"x", "y", "z"}, BoxRatios → {1, 1, 1},
 PlotStyle → {Blue, Thick}]
```

Below is the Python code of figure 2 and 3:

```
import numpy as np
import matplotlib.pyplot as plt

def logistic_map(x, r):
    return r * x * (1 - x)

def generate_bifurcation_diagram(r_values, x0, num_iterations, num_transient, num_points):
    bifurcation_diagram = []

    for r in r_values:
        x = x0
        for _ in range(num_transient):
            x = logistic_map(x, r)

        for _ in range(num_points):
            x = logistic_map(x, r)
            bifurcation_diagram.append([r, x])

    return np.array(bifurcation_diagram)

# Parameters
r_values = np.linspace(0, 4.0, 1000) #change 0 to be 2.5, then it will be figure 2.2
x0 = 0.5
num_iterations = 100
num_transient = 100
num_points = 500

# Generate bifurcation diagram
bifurcation_data = generate_bifurcation_diagram(r_values, x0, num_iterations, num_transient, num_points)

# Plotting
plt.figure(figsize=(10, 6))
plt.scatter(bifurcation_data[:, 0], bifurcation_data[:, 1], s=0.05, color='blue', marker='.')
plt.xlabel('Parameter (rho)')
plt.ylabel('x')
plt.title('Period-Doubling Bifurcation Diagram')
plt.show()
```

Below is the Mathematica code of figure 4, 5 and 6:

```
σ = 10;
ρ = 28;
β = 8 / 3;
lorenz = {
   x'[t] == σ (y[t] - x[t]),
   y'[t] == x[t] (ρ - z[t]) - y[t],
   z'[t] == x[t] y[t] - β z[t]
};
initialCondition1 = {x[0] == 0, y[0] == 1, z[0] == 1};
initialCondition2 = {x[0] == 0.0000000000001, y[0] == 1, z[0] == 1};
sol1 = NDSolve[{lorenz, initialCondition1}, {x, y, z}, {t, 0, 50}];
sol2 = NDSolve[{lorenz, initialCondition2}, {x, y, z}, {t, 0, 50}];
F1 = Plot[Evaluate[x[t] /. sol1], {t, 0, 50}, AxesLabel → {"t", "x"}, PlotStyle → Red];
F2 = Plot[Evaluate[x[t] /. sol2], {t, 0, 50}, AxesLabel → {"t", "x"}];
F3 = Plot[Evaluate[y[t] /. sol1], {t, 0, 50}, AxesLabel → {"t", "y"}, PlotStyle → Red];
F4 = Plot[Evaluate[y[t] /. sol2], {t, 0, 50}, AxesLabel → {"t", "y"}];
F5 = Plot[Evaluate[z[t] /. sol1], {t, 0, 50}, AxesLabel → {"t", "z"}, PlotStyle → Red];
F6 = Plot[Evaluate[z[t] /. sol2], {t, 0, 50}, AxesLabel → {"t", "z"}];
A = Show[F1, F2];
B = Show[F3, F4];
F = Show[F5, F6];
GraphicsColumn[{A, B, F}]
```

Below is the Python code for displaying the value of variables x, y and z with respect to time t:

```
import numpy as np
from scipy.integrate import solve_ivp
import matplotlib.pyplot as plt

# Lorenz system equations
def lorenz(t, xyz, sigma, rho, beta):
    x, y, z = xyz
    dxdt = sigma * (y - x)
    dydt = x * (rho - z) - y
    dzdt = x * y - beta * z
    return [dxdt, dydt, dzdt]

# Set up parameters
sigma = 10
rho = 28
beta = 8 / 3

# Set initial conditions
initial_conditions = [1.0, 1.0, 1.0]

# Set time span
t_span = (0, 25)
t_eval = np.linspace(*t_span, 10000)

# Numerically solve the Lorenz system
solution = solve_ivp(lorenz, t_span, initial_conditions, args=(sigma, rho, beta), t_eval=t_eval)

# Access the solution
t = solution.t
x, y, z = solution.y

# Display the values
for i in range(len(t)):
    print(f"Time: {t[i]}, x: {x[i]}, y: {y[i]}, z: {z[i]}")
```

Declaration

I, **KIMSIE PHAN**, hereby declare that the work presented in this research project, submitted to **Mr. Vishal Dhawan**, Assistant Professor, Department of Mathematics in partial fulfillment of the requirements for applying for the scholarship. I confirm that this project work is my original work and that any work done by others or by myself previously has been acknowledged and referenced accordingly. This work was not previously presented to another examination board and has not been published.

(Signature of Scholar)

Date: 11 December, 2023

Place: Delhi