How To Solve System of Linear Equation Using Matrix Notation

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Definition (Multiplication of Matrix)

If A is an $m \times n$ matrix and B is an $n \times p$ matrix,

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \qquad B = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1p} \\ b_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix}$$

the matrix product $C = AB$ is defined to be the $m \times p$ matrix

$$C = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1p} \\ c_{21} & c_{22} & \cdots & c_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mp} \end{pmatrix}$$

such that $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}$,
for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, p$.

Definition (Systems of Linear Equations)

Suppose F is a field. We consider the problem of finding n scalars (elements of F) $x_1, x_2, ..., x_n$ which satisfy the condition

$$\begin{aligned} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n &= B_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n &= B_2 \\ \vdots \qquad \vdots \qquad &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n &= B_m \end{aligned} \tag{1}$$

where $B_1, ..., B_m$ and $A_{ij}, \quad 1 \leq i \leq m, \ 1 \leq j \leq n$, are given elements of F. We call (1) a **system of m linear equation in n unknowns.**

How to write systems of linear equation to matrix form

Suppose that

$$\begin{cases} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n & = B_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n & = B_2 \\ \vdots \qquad \vdots & \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n & = B_m \end{cases} \tag{2}$$

is the system of m linear equation in n unknowns.
Let

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & . & Amn \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_m \end{bmatrix}$$

Note:

- Make sure that all of the equations are written in a similar manner, meaning the variables need to all be in the same order.
- Make sure that one side of the equation is only variables and their coefficients, and the other side is just constants.

Using matrix multiplication, we may define a system of equations with the same number of equations as variables as:

$$AX = B.$$

To solve a system of linear equations using an inverse matrix,
We have

$$\begin{aligned} AX &= B, && \text{multiple by inverse of matrix } A \\ A^{-1}AX &= A^{-1}B, && \text{multiple between } A^{-1}A = I,\ I \text{ is unity matrix} \\ IX &= A^{-1}B, && \text{multiple between } IX = X \\ X &= A^{-1}B \end{aligned}$$

Therefore, to find answer X, we have to find the inverse of matrix A i.e., the systems of linear equation has answer when the matrix A have inverse of it.

Theorem

Let $AX = B$ be a system of linear equation, where A is the coefficient matrix. If A is invertible then the system has a unique solution, given by

$$X = A^{-1}B.$$

Proof: We have

$$AX = B \Longrightarrow X = A^{-1}B$$

If $AX = B$ has two sets of solution X_1 and X_2, we get

$$AX_1 = B \quad \text{and} \quad AX_2 = B \quad \text{then} \quad AX_1 = AX_2$$

By cancellation law, A is being invertible, then

$$X_1 = X_2$$

Therefore, the system of linear equation $AX = B$ has a unique solution.

How to find the inverse of matrix?

Definition (Elementary row operation)

Three types of elementary row operations can be performed on matrices:

1. Interchanging two rows:
   $R_i \longleftrightarrow R_j$ interchanges rows i and j.
2. Multiplying a row by a nonzero scalar:
   $R_i \longrightarrow tR_i$ multiplies row i by the nonzero scalar t.
3. Adding a multiple of one row to another row:
   $R_j \longrightarrow R_j + tR_i$ adds t times row i to row j.

Definition (Row Equivalent)

Matrix A is row-equivalent to matrix B if B is obtained from A by a sequence of elementary row operations. We denoted

$$A = E_n E_{n-1} ... E_2 E_1 B,$$

where $E_1, E_2, ..., E_n$ are elementary matrices.

Lemma

An elementary matrix is nonsingular and its inverse is an elementary matrix of the same type.

Theorem

An $n \times n$ matrix A is nonsingular if and only if A is a product of elementary matrices.

Corollary

An $n \times n$ matrix A is nonsingular if and only if A is row equivalent to I.

Theorem

Let A be an $n \times n$ matrix and suppose that A is row equivalent to I. Then A is nonsingular, and A^{-1} can be found by performing the same sequence of elementary row operations on I as were used to convert A to I.

Proof: Suppose that A is row equivalent to I then

$$I = E_n E_{n-1} ... E_2 E_1 A$$

where $E_1, E_2, ..., E_n$ are elementary matrices. Then

$$I_n A^{-1} = E_n E_{n-1} ... E_2 E_1 A A^{-1}$$
$$A^{-1} = E_n E_{n-1} ... E_2 E_1$$

We now have an effective algorithm for computing A^{-1}. We use elementary row operations to transform A to I_n; the product of the elementary matrices $E_n E_{n-1} ... E_2 E_1$ gives A^{-1}. The algorithm can be efficiently organized as follows. Form the $n \times n$ matrix $[A|I_n]$ and perform elementary row operations to transform this matrix to $\left[I_n|A^{-1}\right]$. Every elementary row operation that is performed on a row of A is also performed on the corresponding row of I_n.

Example

Find an inverse of matrix $A = \begin{bmatrix} 3 & 1 & 6 \\ 1 & -1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$.

Answer: Find an inverse of matrix $A = \begin{bmatrix} 3 & 1 & 6 \\ 1 & -1 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

Observe,

$$\left[\begin{array}{rrr|rrr} 3 & 1 & 6 & 1 & 0 & 0 \\ 1 & -1 & 4 & 0 & 1 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 4 & 0 & 1 & 0 \\ 3 & 1 & 6 & 1 & 0 & 0 \\ 3 & 2 & -2 & 0 & 0 & 1 \end{array}\right], \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 4 & 0 & 1 & 0 \\ 3 & 1 & 6 & 1 & 0 & 0 \\ 0 & -1 & 8 & 1 & 0 & -1 \end{array}\right], \quad R_3 \rightarrow R_2 - R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 4 & 0 & 1 & 0 \\ 0 & 4 & -6 & 1 & -3 & 0 \\ 0 & -1 & 8 & 1 & 0 & -1 \end{array}\right], \quad R_2 \to -3R_1 + R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -1 & 4 & 0 & 1 & 0 \\ 0 & 4 & -6 & 1 & -3 & 0 \\ 0 & 0 & 26 & 5 & -3 & -4 \end{array}\right], \quad R_3 \to R_2 + 4R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & \frac{5}{2} & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & 4 & -6 & 1 & -3 & 0 \\ 0 & 0 & 1 & \frac{5}{26} & -\frac{3}{26} & -\frac{2}{13} \end{array}\right], \quad R_3 \to \frac{1}{26}R_3,\ R_1 \to R_1 + \frac{1}{4}R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{3}{13} & \frac{7}{13} & \frac{5}{13} \\ 0 & 1 & -\frac{3}{2} & \frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & 0 & 1 & \frac{5}{26} & -\frac{3}{26} & -\frac{2}{13} \end{array}\right], \quad R_1 \to R_1 - \frac{5}{2}R_3,\ R_2 \to \frac{1}{4}R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{3}{13} & \frac{7}{13} & \frac{5}{13} \\ 0 & 1 & 0 & \frac{7}{13} & -\frac{12}{13} & -\frac{3}{13} \\ 0 & 0 & 1 & \frac{5}{26} & -\frac{3}{26} & -\frac{2}{13} \end{array}\right], \quad R_2 \to R_2 + \frac{3}{2}R_3$$

Therefore, $A^{-1} = \begin{bmatrix} -\frac{3}{13} & \frac{7}{13} & \frac{5}{13} \\ \frac{7}{13} & -\frac{12}{13} & -\frac{3}{13} \\ \frac{5}{26} & -\frac{3}{26} & -\frac{2}{13} \end{bmatrix}$

Example

Solve the system of linear equations

$$\begin{cases} 2x_1 + 3x_2 + 3x_3 & = 5 \\ x_1 - 2x_2 + x_3 & = -4 \\ 3x_1 - x_2 - 2x_3 & = 3 \end{cases}$$

Answer: Solve the system of linear equations $\begin{cases} 2x_1 + 3x_2 + 3x_3 & = 5 \\ x_1 - 2x_2 + x_3 & = -4 \\ 3x_1 - x_2 - 2x_3 & = 3 \end{cases}$

Let

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

Then

$$AX = B \Longrightarrow X = A^{-1}B$$

Now,

$$\left[\begin{array}{rrr|rrr} 2 & 3 & 3 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 3 & -1 & -2 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 0 & 1 & 0 \\ 2 & 3 & 3 & 1 & 0 & 0 \\ 3 & -1 & -2 & 0 & 0 & 1 \end{array}\right], \quad R_1 \leftrightarrow R_2$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 7 & 1 & 1 & -2 & 0 \\ 0 & 5 & -5 & 0 & -3 & 1 \end{array}\right], \quad R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 7 & 1 & 1 & -2 & 0 \\ 0 & 0 & 40 & 5 & 11 & -7 \end{array}\right], \quad R_3 \to 5R_2 - 7R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & \frac{1}{7} & \frac{1}{7} & -\frac{2}{7} & 0 \\ 0 & 0 & 1 & \frac{1}{8} & \frac{11}{40} & -\frac{7}{40} \end{array}\right], \quad R_2 \to \frac{1}{7}R_2,\ R_3 \to \frac{1}{40}R_3$$

$$\left[\begin{array}{rrr|rrr} 1 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \frac{1}{8} & -\frac{13}{40} & \frac{1}{40} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{11}{40} & -\frac{7}{40} \end{array}\right], \quad R_2 \to R_2 - \frac{1}{7}R_3$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{4} & \frac{7}{20} & \frac{1}{20} \\ 0 & 1 & 0 & \frac{1}{8} & -\frac{13}{40} & \frac{1}{40} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{11}{40} & -\frac{7}{40} \end{array}\right], \quad R_1 \to R_1 + 2R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{8} & \frac{5}{8} & -\frac{1}{8} \\ 0 & 1 & 0 & \frac{1}{8} & -\frac{13}{40} & \frac{1}{40} \\ 0 & 0 & 1 & \frac{1}{8} & \frac{11}{40} & -\frac{7}{40} \end{array}\right], \quad R_1 \to R_1 - R_3$$

Thus

$$A^{-1} = \begin{bmatrix} \frac{1}{8} & \frac{5}{8} & -\frac{1}{8} \\ \frac{1}{8} & -\frac{13}{40} & \frac{1}{40} \\ \frac{1}{8} & \frac{11}{40} & -\frac{7}{40} \end{bmatrix} = \frac{1}{8}\begin{bmatrix} 1 & 5 & -1 \\ 1 & -\frac{13}{5} & \frac{1}{5} \\ 1 & \frac{11}{5} & -\frac{7}{5} \end{bmatrix}$$

Then

$$X = \frac{1}{8}\begin{bmatrix} 1 & 5 & -1 \\ 1 & -\frac{13}{5} & \frac{1}{5} \\ 1 & \frac{11}{5} & -\frac{7}{5} \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{8}\begin{bmatrix} -18 \\ 16 \\ -8 \end{bmatrix}$$

Therefore,

$$x_1 = -\frac{9}{4},\ x_2 = 2,\ x_3 = -1.$$

References

1. **Kenneth Hoffman**, *Linear Algebra*. Second Edition, Prentice-hall, Inc., Englewood Cliffs, New Jersey. 1971.
2. **Stephen H. Friedberg**, *Linear Algebra*. Fourth Edition, Pearson Education, Inc. 2003.